HOSSAM GHANEM

(7) 7.4 Integral Using Natural Logarithm and Exponential Function (B)

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

Example 1 Prove that $\ln pq = \ln p + \ln q$ 25 April 2008

Solution

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

$$\ln pq = \int_1^{pq} \frac{1}{t}\, dt = \int_1^{p} \frac{1}{t}\, dt + \int_p^{pq} \frac{1}{t}\, dt$$

$$I_1 = \int_p^{pq} \frac{1}{t}\, dt$$

Let $ps = t$ $\therefore p\ ds = dt$

at $t = p$ $\rightarrow$ $ps = p$ $\rightarrow$ $s = 1$

at $t = pq$ $\rightarrow$ $ps = pq$ $\rightarrow$ $s = q$

$$I_1 = \int_1^{q} \frac{1}{ps}\, p\ ds = \int_1^{q} \frac{1}{s}\, ds$$

$$\therefore \ln pq = \int_1^{p} \frac{1}{t}\, dt + \int_1^{q} \frac{1}{s}\, ds = \ln p + \ln q$$

Example 2 Prove that $\ln \frac{p}{q} = \ln p - \ln q$

Solution

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

$$\ln\left(\frac{p}{q}\right) = \int_1^{p/q} \frac{1}{t} dt = \int_1^{p} \frac{1}{t} dt + \int_p^{p/q} \frac{1}{t} dt$$

$$I_1 = \int_p^{p/q} \frac{1}{t} dt$$

Let $\frac{p}{q}\, s = t$ $\frac{p}{q}\, ds = dt$

at $t = p$ $\frac{p}{q}\, s = p$ $s = q$

at $t = \frac{p}{q}$ $\quad \frac{p}{q}\, s = {}^{p}/_{q}$ $\quad s = 1$

$$I_1 = \int_q^1 \frac{q}{ps} \cdot \frac{p}{q} ds = \int_q^1 \frac{1}{s} ds = -\int_1^q \frac{1}{s} ds$$

$$\ln\left(\frac{p}{q}\right) = \int_1^p \frac{1}{t} dt - \int_p^q \frac{1}{s} ds = \ln p - \ln q$$

Example 3 Prove that $\ln p^r = r \ln p$

10 March 1999
19 March 2006 A

Solution

$\ln(x) = \int_1^x \frac{1}{t} dt$

$$\ln(p^r) = \int_1^{p^r} \frac{1}{t} dt$$

Let $s^r = t$ $\quad rs^{r-1} ds = dt$

$at \quad t = p^r \quad s^r = p^r \quad s = p$

$at \quad t = 1 \quad s^r = 1 \quad s = 1$

$$\ln(p^r) = \int_1^p \frac{1}{s^r} \cdot rs^{r-1} ds = r \int_1^p \frac{1}{s} ds = r \ln p$$

Example 4 Prove that $\ln \frac{1}{p} = -\ln p$

29 July 2009 A

Solution

$$\ln x = \int_1^x \frac{1}{t} dt$$

$$\ln \frac{1}{p} = \int_1^{1/p} \frac{1}{t} dt$$

Let $\frac{1}{s} = t$ $\quad \therefore \frac{-1}{s^2} ds = dt$

$at \quad t = 1 \quad \frac{1}{s} = 1 \quad s = 1$

$at \quad t = \frac{1}{p} \quad \frac{1}{s} = \frac{1}{p} \quad s = p$

$$\ln \frac{1}{p} = \int_1^p s \cdot \frac{-1}{s^2} ds = -\int_1^p \frac{1}{s} ds = -\ln p$$

Example 5 Find $\int \sec x \ dx$

Solution

$$I = \int \sec x \ dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} \ dx$$ بالضرب في $(\sec x + \tan x)$ بسط ومقام

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} \ dx$$

Let $t = \sec x + \tan x$ $\quad dt = (\sec x \cdot \tan x + \sec^2 x)dx$

$$I = \int \frac{1}{t} dt = \ln|t| + c = \ln|\sec x + \tan x| + c$$

Example 6
24 March 2008 A . Evaluate the following integrals $\int \frac{\sec x}{\sin x + 4\cos x} dx.$

Solution

$$I = \int \frac{\sec x}{\sin x + 4\cos x} dx = \int \frac{\sec^2 x}{\tan x + 4} dx$$ بالضرب في $(\sec x)$ بسط ومقام

Let $t = \tan x + 4$ $\quad dt = \sec^2 x \, dx$

$$\therefore I = \int \frac{1}{t} \ dt = \ln|t| + c = \ln|\tan x + 4| + c$$

Example 7
5 October 1996 . Evaluate the following integrals $\int \left(\frac{1}{\sec(5x)} + \sec(2-3x)\right) dx$

Solution

$$I = \int \left(\frac{1}{\sec(5x)} + \sec(2-3x)\right) dx = \int (\cos(5x) + \sec(2-3x)) \ dx$$

$$= \frac{1}{5}\sin 5x + \frac{-1}{3}\ln|\sec(2-3x) + \tan(2-3x)| + c = \frac{1}{5}\sin 5x - \frac{1}{3}\ln|\sec(2-3x) + \tan(2-3x)|$$

Example 8
9 October 1998 . Evaluate the following integrals $\int (1 + \csc x)^2 \, dx;$

Solution

$$I = \int (1 + \csc x)^2 \, dx = \int (1 + 2\csc x + \csc^2 x) \, dx = x + 2\ln|\csc x - \cot x| - \cot x + c$$

Example 9 Find $\int \cot x \ dx$

Solution

$$I = \int \cot x \ dx = \int \frac{\cos x}{\sin x} dx$$

Let $t = \sin x$ $\quad dt = \cos x \, dx$

$$I = \int \frac{1}{t} dt = \ln|t| + c = \ln|\sin x| + c$$

Example 10
13 March 2001 A

. Evaluate the following integrals $\int \frac{\sec(2^{-x})}{2^x}dx$

Solution

$I = \int \frac{\sec(2^{-x})}{2^x}dx$

Let $t = 2^{-x}$ $dt = 2^{-x}\ln 2\,(-1)dx$ $\frac{-1}{\ln 2}dt = \frac{1}{2^x}dx$

$I = \frac{-1}{\ln 2}\int \sec t\,dt = \frac{-1}{\ln 2}\ln|\sec t + \tan u| + c = \frac{-1}{\ln 2}\ln|\sec(2^{-x}) + \tan(2^{-x})| + c$

Example 11
34 July 9, 2011

(3 pts) Evaluate the following integrals. $\int \frac{dx}{(1+e^{-x})\ln(1+e^x)}$

Solution

$I = \int \frac{dx}{(1+e^{-x})\ln(1+e^x)}$

بالضرب في e^{-x}

Let $t = \ln(1+e^x)$ $dt = \frac{e^x}{1+e^x}dx$ $dt = \frac{1}{1+e^{-x}}dx$

$I = \int \frac{1}{\ln(1+e^x)}\cdot\frac{1}{(1+e^{-x})}dx = \int \frac{1}{t}dt = \ln|t| + c = \ln|\ln(1+e^x)| + c$

Example 12
14 March 2002

. Evaluate the following integrals $\int \frac{\cot x}{2+\ln(\sin^3 x)}\,dx$

Solution

$I = \int \frac{\cot x}{2+\ln(\sin^3 x)}\,dx = \int \frac{\cot x}{2+3\ln(\sin x)}\,dx$

$t = 2 + 3\ln(\sin x)$ $dt = 3\frac{\cos x}{\sin x}dx$ $\frac{1}{3}dt = \cot x\,dx$

$I = \frac{1}{3}\int \frac{1}{t}dt = \frac{1}{3}\ln|t| + c = \frac{1}{3}\ln|2 + 3\ln(\sin x)| + c$

Homework

1	Evaluate the following integral	$\int (\tan x + 1)^2 \, dx$	SECOND SEMESTER 87/88
2	Evaluate the following integral	$\int \frac{\sec(x)}{\cos(x) + \sin(x)} \, dx$	21 March 2007 A
3	Evaluate the following integral	$\int \csc x \, (1 - \csc x) \, dx$	
4	Evaluate the following integral	$\int \frac{\cot(e^{-3x})}{e^{3x}} \, dx$	9 October 1998
5	Evaluate the following integrals	$\int \frac{\sec(\ln x)}{x} \, dx$	7 July 1997
6	Evaluate the following integral	$\int \frac{3 - 2\tan x}{2 + 3\tan x} dx$	13 March 2001 A
7	Evaluate the following integral	$\int (\tan(1 - 2x) + x \sec x^2) \, dx$	
8	Evaluate the following integral	$\int \tan[\ln(x^x e^x)] \ln\left(\frac{1}{xe^2}\right) dx$	18 July 2005 A
9	Evaluate the following integral	$\int \csc\left(\frac{1}{2}x\right) dx$	
10	Evaluate the following integral	$\int \csc x \, dx$	
11	Evaluate the following integral	$\int (\cos x) \, e^{\ln(2 - \sin^2 x)} \, dx$	SECOND SEMESTER 87/88
12	Evaluate the following integral	$\int \tan x \, dx$	
13	Evaluate the following integral	$\int \frac{\sec^2 x}{4 + 3\tan x} \, dx$	30 April 11, 2010
14	Evaluate the integral	$\int \frac{(\ln x)^3}{x\sqrt{4 - (\ln x)^4}} dx$	43 May 2007 A

8
18 July 2005 A . Evaluate the following integrals $\int \tan[\ln(x^x e^x)] \ln\left(\frac{1}{xe^2}\right) dx$

Solution

$t = \ln(x^x e^x)$ $t = x \ln xe$ $du = \left(\ln xe + \frac{xe}{xe}\right) dx$

$du = (\ln xe + 1)dx$

$du = (\ln xe + \ln e)dx$

$du = \ln(xe^2)\ dx$

$du = -\ln\frac{1}{xe^2}\ dx$

$I = -\int \tan u\ du$

$= \ln|\cos u| + c$

$= \ln|\cos \ln(x^x e^x)| + c$